IMM-NYU 164
CL-HU 29

MATHEMATICAL DETERMINATION OF TRANSIENT TEMPERATURES IN SOLIDS

By

EUGENE ISAACSON, New York University
J. ORTEN GADD, Harvard University
THEODORE SINGER, Harvard University

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.

IMM-164
c.1

March, 1948

IMM-NYU 164
CL-HU 29

Mathematical Determination of Transient Temperatures in Solids

Eugene Isaacson, New York University
J. Orten Gadd, Harvard University
Theodore Singer, Harvard University

This report was prepared for the Office of Naval Research by the Institute for Mathematics and Mechanics of New York University, Contract No. N6ori-201, Task Order No. 1, in cooperation with the Computation Laboratory of Harvard University, Contract No. NOrd 8555.

Richard Courant
Director of Research
Institute for Mathematics
and Mechanics
New York University

Howard H. Aiken
Director of Research
Computation Laboratory
of Harvard University

Preface

The problem solved in this report arose out of the study of rockets. It was suggested by Dr. J. K. L. MacDonald of the Institute for Mathematics and Mechanics at New York University. His advice and that of Dr. Samuel Schaaf contributed greatly to the simplification of its analytical aspects, which was the necessary preparation for the numerical work. This latter work was carried out by the teamwork of the staff of the Computation Laboratory at Harvard University on the Mark I computer.

The report may be considered as typical of the possibilities for numerical solutions of complex mathematical problems, if analytical skill is integrated with the methods opened up by the development of automatic computers.

Howard H. Aiken
R. Courant

Mathematical Determination of Transient Temperatures in Solids

Introduction

We investigate the problem of determining mathematically the difference between the temperature which is read on a thermocouple that is inserted into the wall of a rocket and the temperature which would occur if the thermocouple were not present.

We find (for the case we treat) that the disturbance which is created by the insertion of the thermocouple is of the order of 30 percent initially, but that this disturbing effect diminishes quickly with time. This evaluation of the perturbation produced by the thermocouple is based on the results of two different methods of analyzing the problem of heat conduction in solids. One method involves a direct numerical computation which necessitates large-scale automatic machine computation; the other method involves the replacement of our original problem by simpler physical problems for which relatively simple mathematical solutions can be found and evaluated with the use of hand computing machines.

In Sections 8 and 9 we point out the close agreement of the results of the two different methods of attacking the problem. We may expect that a slight variation of the physical parameters will not materially affect the good approximation which was obtained by studying the simpler physical problems for which mathematical solutions could easily be found and evaluated. Thus, the necessity of repeating the lengthy computation for other parameters is removed and a better understanding of the entire

problem achieved. In our case, as presumably in many others, the effectiveness of large-scale automatic machine computation is greatly improved by intimately coupling it with analytical methods specifically adapted to the concrete problem.

In Sections 1-7 we formulate the mathematical problems and the numerical procedures which can be used to evaluate the solutions of these problems.

In Section 10 we present a description of the techniques that were employed to make the large-scale automatic machine computation feasible on the Mark I. In addition an analysis of the pertinent operational and numerical procedures is given which would be of use in the future should other problems of this type be tackled.

Sec. 1 Statement of problem

In order to measure the temperature in the steel wall of a rocket, a thin cylindrical thermocouple is inserted into a hole that was drilled through the outer surface of the wall. The conductivity and the heat capacitance of the various parts of the thermocouple are different from those of steel and consequently the temperatures which are measured are not those which would obtain in the wall when the thermocouple is absent. The purpose of this report is to mathematically determine the difference between the measured temperature and the actual temperature.

We consider a simplified physical model. A steel cylinder of radius 5L and height 5L has a concentric hole of radius L and depth 4L. Into this hole a thermocouple of radius L and height 4L is inserted. The thermocouple is composed of a nonconducting cylindrical outer shell of thickness $\frac{L}{2}$ and a conducting cylindrical core of radius $\frac{L}{2}$ (Figure 1). We assume that, except for the hot wall of the rocket, all of the outer surface is nonconducting.

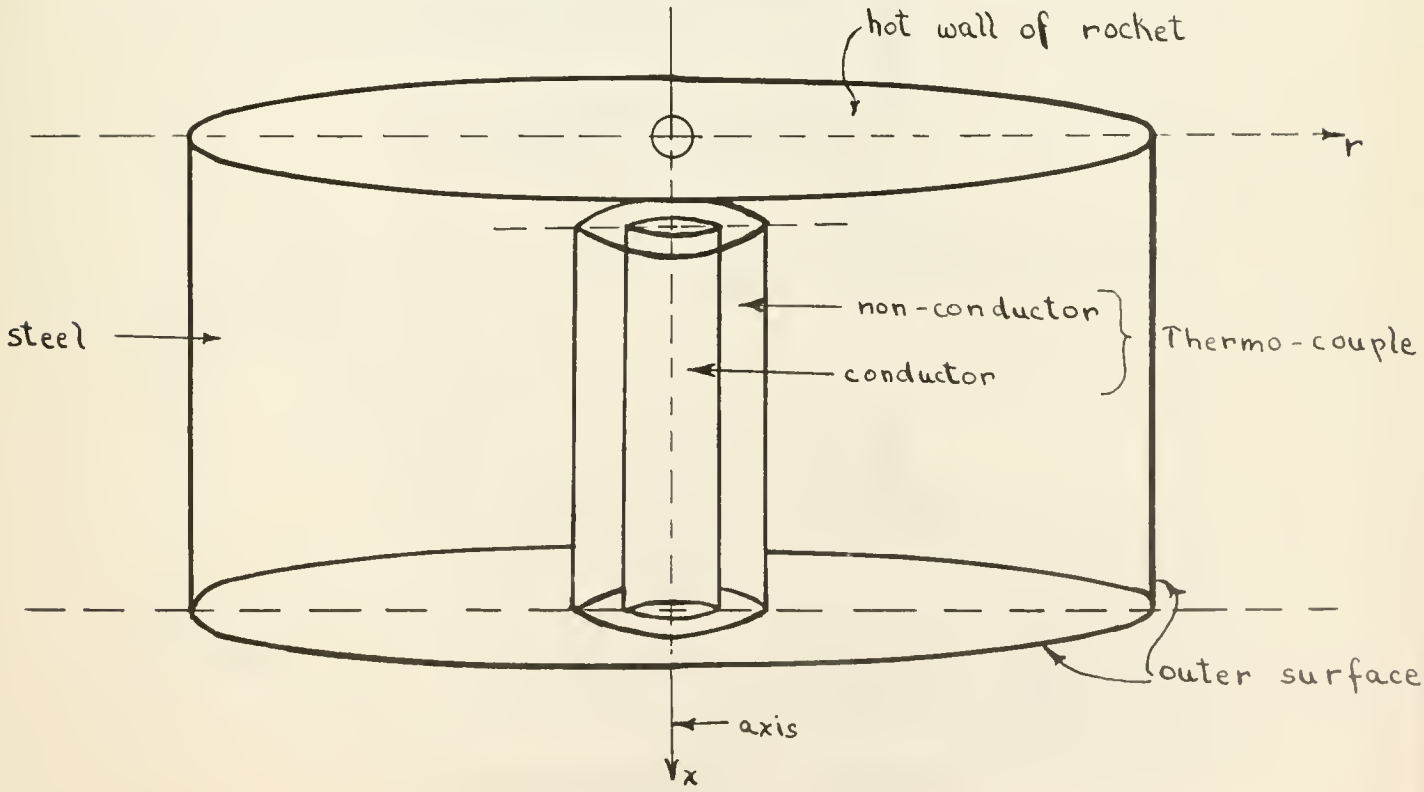

Figure 1

Sec. 2 Mathematical formulation of the problem

The rocket gases heat the plane wall, x = 0, for a period of time of the order of .1 second. We consider a plane section taken through the axis, r = 0, of the solid (Figure 2). The steel portion is called Region I. The part occupied by the constantin core of the thermocouple is called Region II.

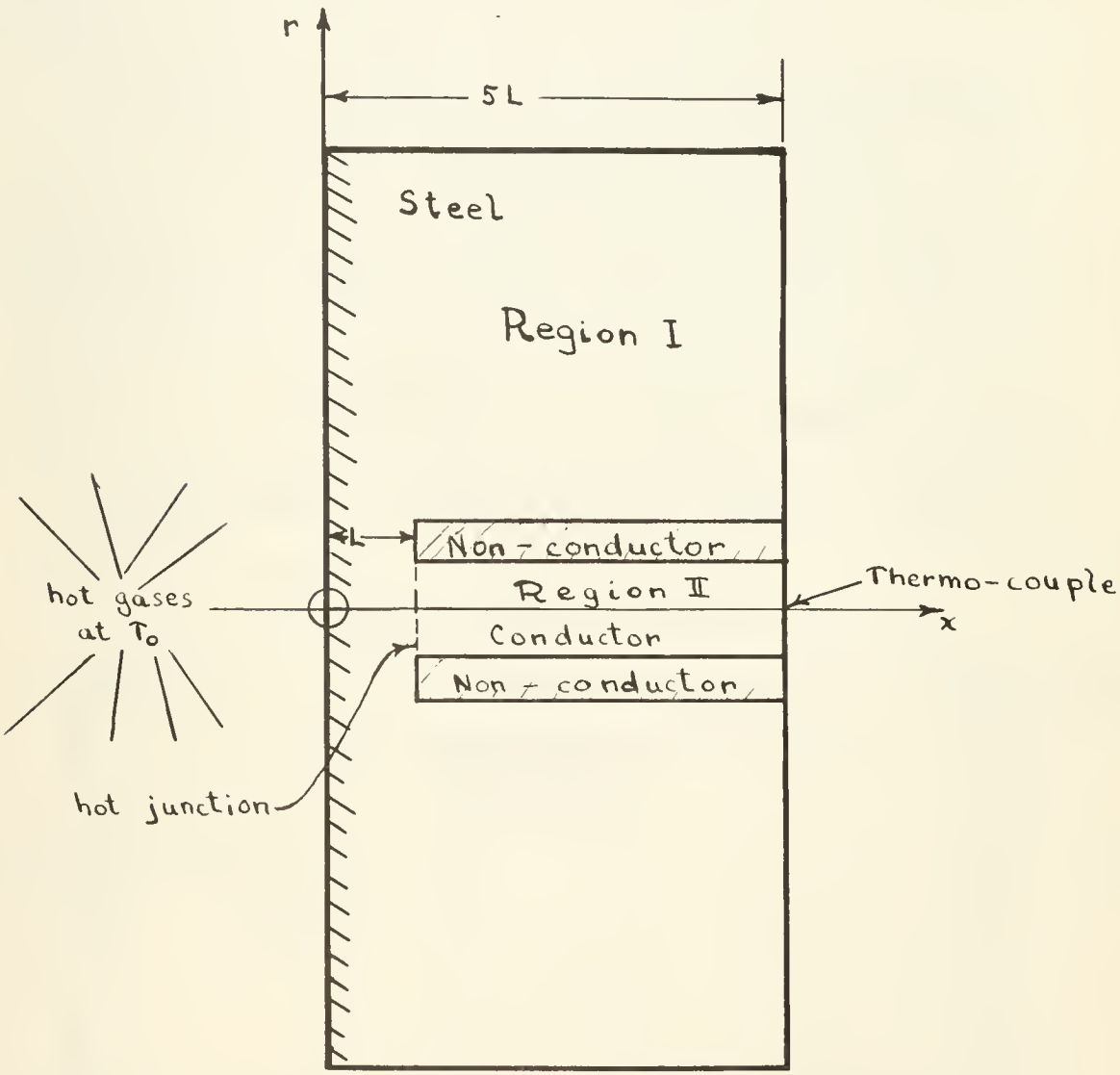

Figure 2

If the thermocouple were not present, the formulas describing the flow of heat would be:

(2.1) I. Partial Differential Equation:

$$k_1 T_{xx} = c_1 T_t,$$

where $T(x,t)$ = temperature ^{o} C,
t = time,
x = distance along axis of cylinder,
k_1 = heat conductivity of steel (constant),
c_1 = heat capacity of steel (constant).

II. Boundary Conditions:

$$k_1 T_x = h_1(T - T_o) \quad \text{at} \quad x = 0,$$

where h_1 = emissivity of steel (constant)
and T_o = temperature of hot gases (constant),

$$T_x = 0 \quad \text{at} \quad x = 5L.$$

III. Initial Conditions:

$$T = 0 \quad \text{for} \quad 0 \leq x \leq 5L, \quad \text{at} \quad t = 0.$$

IV. Constants--c.g.s. units:

$$T_o = 3000^{o} \text{ C},$$
$$L = \frac{.022}{3},$$
$$h_1 = .18, \quad k_1 = .11, \quad c_1 = .84 \quad \text{(for steel)}.$$

On the other hand, when the thermocouple is present, the equations are:

(2.2) I. P.D.E.:

$$k[T_{xx} + T_{rr} + \frac{1}{r} T_r] = cT_t,$$

where $T(x,r,t)$ = temperature ^{o} C,
x = distance along axis of thermocouple,
r = radial distance from axis of thermocouple,
k and c refer to steel in Region I and constantin in Region II.

II. B.C.:

(a) $kT_x = h(T - T_o)$ at $x = 0$,

(b) the normal derivative (T_x or T_r) of T vanishes across the sides of the nonconductor,

(c) $T_x = 0$ at $x = 5L$,

(d) $T_r = 0$ at $r = 5L$,

(e) T and kT_x are continuous across hot junction (the k of steel at the left and the k of constantin at the right).

III. I.C.:

$$T(x,r,0) = 0.$$

IV. Constants--c.g.s. units:

For constantin core of thermocouple

$$k_2 = .055, \quad c_2 = .84.$$

We should solve problems (2.1) and (2.2) and compare the temperature at the hot junction in (2.2) with the temperature at x = L in (2.1). The ratio of these temperatures will determine the magnitude of the factor by which we may expect the thermocouple readings to exceed the actual temperature.

This completes the mathematical formulation of our problem and we proceed to the solution.

Sec. 3 Introduction of dimensionless quantities

We find it convenient to simplify (2.1) and (2.2) by setting

$$(3.1)\quad u = \frac{T}{3000}\ ; \quad \xi = \frac{h_1}{k_1}\,x, \quad \eta = \frac{h_1}{k_1}\,r; \quad \tau = \frac{h_1^2}{k_1 c_1}\,t$$

(h_1, k_1, c_1 of steel). In these dimensionless units the length $\ell = \frac{h_1}{k_1} L = .012$.

Problem (2.1) becomes:

$$(3.2)\quad \text{P.D.E.} \qquad u_{\xi\xi} = u_\tau,$$

$$(3.3)\quad \text{B.C.} \qquad u_\xi = u - 1 \quad \text{at} \quad \xi = 0,$$

$$u_\xi = 0 \quad \text{at} \quad \xi = 5\ell,$$

$$(3.4)\quad \text{I.C.} \qquad u(\xi,0) = 0.$$

The explicit "Fourier series" solution of this problem is found by setting $u = 1 + w$, and using a superposition of fundamental solutions w of the form $w = f(\xi)g(\tau)$. Whence,

$$(3.5)\qquad u = 1 + \sum_{n=1}^{\infty} b_n X_n(\xi) e^{-d_n^2}, \quad \text{where}$$

d_n is the n^{th} positive root of

$$d_n = \cot 5\ell d_n,$$

$$X_n(\xi) = d_n \cot d_n\xi + \sin d_n\xi, \quad \text{and}$$

$$b_n = -\frac{\int_0^{5\ell} X_n(\xi)\,d\xi}{\int_0^{5\ell} X_n^2(\xi)\,d\xi} = -\frac{2}{d_n[1 + 5\ell(1 + d_n^2)]}\ .$$

Problem (2.2) becomes:

$$\text{(3.6)}\quad \underline{\text{P.D.E.}} \quad \begin{cases} u_{\xi\xi} + u_{\eta\eta} + \frac{1}{\eta}\, u_\eta = u_\tau; & \text{in Region I,} \\ u_{\xi\xi} + u_{\eta\eta} + \frac{1}{\eta}\, u_\eta = \frac{c_2}{k_2}\,\frac{k_1}{c_1}\, u_\tau; & \text{in Region II.} \end{cases}$$

$$\text{(3.7)}\quad \underline{\text{B.C.}} \quad \begin{cases} \text{(a)} & u_\xi = u - 1, \\ \text{(b)} & \text{the normal derivative } (u_\xi \text{ or } u_\eta) \text{ vanishes across the sides of the nonconductor,} \\ \text{(c)} & u_\xi = 0 \quad \text{at} \quad \xi = 5\ell, \\ \text{(d)} & u_\eta = 0 \quad \text{at} \quad \eta = 5\ell, \\ \text{(e)} & u_{\text{left}} = u_{\text{right}}, \\ & k_1 u_{\xi\,\text{left}} = k_2 u_{\xi\,\text{right}}. \end{cases}$$

$$\text{(3.8)}\quad \underline{\text{I.C.}} \qquad u(\xi,\eta,0) = 0.$$

If we could find the solution of problem (3.6) in a form permitting the calculation of the temperature at the "hot junction," our task would be simple. However, we cannot expect to obtain as simple a solution of (3.6) as we found for (3.2). We therefore attempt a direct numerical solution of (3.6) by the method of finite differences.

Sec. 4 The finite difference equations

Since the domain of our problem (Figure 2) is symmetric about the $\eta = 0$ axis, we need only consider the solution of (3.6) for $0 \le \xi \le 5\ell$ and $0 \le \eta \le 5\ell$. We replace derivatives in (3.6) and (3.7) by their corresponding difference quotients and use the notation

$u^k_{i,j}$	to represent the value of u at ξ_i, η_j, τ_k,
$\Delta\xi$, $\Delta\eta$, and $\Delta\tau$	to mean the intervals in ξ, η, and τ.

(4.1) At non-boundary points in Region I, the partial differential equation is replaced by:

$$\frac{u^o_{-1,o}-2u^o_{o,o}+u^o_{1,o}}{(\Delta\xi)^2}+\frac{u^o_{o,-1}-2u^o_{o,o}+u^o_{o,1}}{(\Delta\eta)^2}+\frac{1}{\eta_o}\,\frac{[u^o_{o,1}-u^o_{o,-1}]}{2\Delta\eta}$$

$$=\frac{u^1_{o,o}-u^o_{o,o}}{\Delta\tau}\,.$$

Therefore,

$$u^1_{o,o}=u^o_{o,o}+\frac{\Delta\tau}{(\Delta\xi)^2}[u^o_{-1,o}+u^o_{1,o}+u^o_{o,-1}+u^o_{o,1}-4u^o_{o,o}]$$

$$+\frac{\Delta\eta\,\Delta\tau}{2\eta_o(\Delta\eta)^2}[u^o_{o,1}-u^o_{o,-1}].$$

We now pick $\Delta\xi = \Delta\eta = \frac{\ell}{2n}$ (n any integer), and $\Delta\tau = \frac{1}{4}(\Delta\xi)^2$. Whence

$$u^1_{o,o}=\tfrac{1}{4}[u^o_{-1,o}+u^o_{1,o}+u^o_{o,-1}+u^o_{o,1}]+\frac{\Delta\eta}{8\eta_o}[u^o_{o,1}-u^o_{o,-1}].$$

(4.2) At non-boundary points in Region II, we obtain in a similar way

$$u^{o}_{o,o} + u^{o}_{o,o} + \frac{\alpha}{4}[u^{o}_{-1,o} + u^{o}_{1,o} + u^{o}_{o,-1} + u^{o}_{o,1} - 4u^{o}_{o,o}]$$

$$+ \frac{\alpha\Delta\eta}{8\eta_o}[u^{o}_{o,1} - u^{o}_{o,-1}],$$

where $\alpha = \frac{k_2 c_1}{k_1 c_2} = \frac{1}{2}$ (in our case).

(4.3) At boundary points on the line $\eta = 0$, we use the symmetry

$$u^{o}_{o,-1} = u^{o}_{o,1}$$

and the fact that the partial differential equations are of the form

$$u_{\xi\xi} + 2u_{\eta\eta} = u_{\tau}, \qquad \text{in Region I;}$$

$$u_{\xi\xi} + 2u_{\eta\eta} = \frac{1}{\alpha} u_{\tau}, \qquad \text{in Region II.}$$

Therefore, in Region I:

$$u^{1}_{o,o} = u^{o}_{o,1} + \frac{1}{4}[u^{o}_{-1,o} - 2u^{o}_{o,o} + u^{o}_{1,o}];$$

while in Region II:

$$u^{1}_{o,o} = \alpha u^{o}_{o,1} + (1 - \alpha)u^{o}_{o,o} + \frac{\alpha}{4}[u^{o}_{-1,o} - 2u^{o}_{o,o} + u^{o}_{1,o}].$$

(4.4) At boundary points on the line $\xi = 0$ (except for corner points), we replace the boundary condition $u_{\xi} = u - 1$ by the corresponding finite difference relation to obtain

$$u^{o}_{o,o} + \Delta\xi\left[\frac{u^{o}_{o,o} - 1}{2} + \frac{u^{o}_{2,o} - u^{o}_{o,o}}{4\Delta\xi}\right] = u^{o}_{1,o}.$$

(We use the fact that the quantity in brackets [ ] corresponds to u_{ξ} at $u^{o}_{\frac{1}{2},o}$.) Therefore,

$$u^{o}_{o,o} = \frac{u^{o}_{1,o} - \frac{1}{4}\, u^{o}_{2,o} + \frac{\Delta\xi}{2}}{\frac{3}{4} + \frac{\Delta\xi}{2}} .$$

(4.5) At boundary points where either $u_\xi = 0$ or $u_\eta = 0$ (except for points at corners), we proceed in a similar way as in (4). For example, on the line $\eta = \ell$

$$u^{o}_{o,o} + \Delta\eta\,[0 + \frac{u^{o}_{o,2} - u^{o}_{o,o}}{4\Delta\eta}] = u^{o}_{o,1}.$$

Whence

$$u^{o}_{o,o} = \frac{4}{3}[u^{o}_{o,1} - \frac{u^{o}_{o,2}}{4}].$$

(4.6) At the "hot junction," $\xi = \ell$, we find

$$u_\xi \text{ on left} = -\frac{1}{2\Delta\xi}[4u^{o}_{-1,o} - 3u^{o}_{o,o} - u^{o}_{-2,o}],$$

while

$$u_\xi \text{ on right} = \frac{1}{2\Delta\xi}[4u^{o}_{1,o} - 3u^{o}_{o,o} - u^{o}_{2,o}].$$

Therefore,

$$u^{o}_{o,o} = \frac{\frac{k_2}{k_1}[4u^{o}_{1,o} - u^{o}_{2,o}] + [4u^{o}_{-1,o} - u^{o}_{-2,o}]}{3(1 + \frac{k_2}{k_1})} .$$

(4.7) At corner points, we use the average of the values which are obtained from the pertinent previously described formulas.

Sec. 5 Solution of finite difference equations

The method of carrying out the solution of (3.6), which is described in Section 4, is entirely straightforward. We set $u = 0$, initially, and we find that heat will flow in across the boundary $\xi = 0$. The magnitude of this job can only be appreciated when one realizes that this is the type of calculation which can only be performed by one person. Consequently, we feel that without the use of an automatic calculator this problem would have to be given up as unsolvable. A complete description of how the Mark I computed the solution of (3.6) is given in Section 10.

But, before such a large-scale computation is attempted there are two questions which must be asked:

(a) At what rate will computational errors propagate and grow? By errors, we mean isolated accidental errors as well as the systematically introduced roundoff errors which arise because we use a fixed number of decimal places.

(b) How small should the interval $\Delta\xi$ be chosen in order that a reasonable degree of approximation is attained and that the amount of labor is not prohibitively large?

In Section 6 we answer (a) and in Section 7 we answer (b).

Sec. 6 Propagation of errors

Because our differential equations and boundary conditions are linear, it is convenient to consider the introduction of errors in the following way. The function u^* which is obtained in the course of the computation may be written as

$$u^*(\xi,\eta,\tau) = u(\xi,\eta,\tau) + \varepsilon(\xi,\eta,\tau), \tag{6.1}$$

where u is the exact solution and ε is the error. Therefore, we know that ε satisfies the same partial differential equations which u and u^* satisfy. In addition, ε satisfies the same boundary conditions which u and u^* satisfy, except the one at $\xi = 0$. At $\xi = 0$, ε satisfies

$$\varepsilon_\xi = \varepsilon, \tag{6.2}$$

a homogeneous condition. This means physically that the temperature represented by ε is being dissipated at $\xi = 0$. We can now find a "Fourier series" solution for ε by seeking fundamental solutions of (3.6) with the above homogeneous boundary condition (6.2).

That is we set

$$\varepsilon(\xi,\eta,\tau) = h(\xi,\eta)k(\tau). \tag{6.3}$$

Then h is a solution of the equation

$$h_{\xi\xi} + h_{\eta\eta} + m^2 h = 0, \tag{6.4}$$

while k satisfies

$$\frac{dk}{d\tau} + m^2 k = 0. \tag{6.5}$$

Equation (6.4) represents a complicated eigen-value problem. That is, there are only certain values of m^2 for

which (6.4) has a non-trivial solution.* We consider all of these solutions $h_m(\xi,\eta)$ of (6.4) and the corresponding solutions $k_m(\tau) = e^{-m^2(\tau-\tau_o)}$ of (6.5). We then set

$$\varepsilon(\xi,\eta,\tau) = \sum_{m=1}^{\infty} u_m h_m(\xi,\eta) e^{-m^2(\tau-\tau_o)}, \tag{6.6}$$

and determine the values of the constants u_m so as to satisfy the "initial" condition. (That is, that at some time τ_o an error is introduced.)

We see from the form of the solution (6.6) that ε will die out exponentially in the time. This corroborates our physical intuition which told us that the temperature ε acts as if it is being radiated at $\xi = 0$ into a medium at 0 temperature.

Consequently, we have demonstrated the computational "stability" of our procedure. The fact that errors are wiped out quickly will be verified when we examine the results of the computations in Section 8.

---

*The existence of eigen-values m^2 has not as yet been established for such a complicated problem.

Sec. 7 Mesh-width

In order to decide upon the size of the mesh-width, we had to take into account the fact that if the space mesh, $\Delta\xi = \Delta\eta$, were halved, the entire job would be 16 times larger. On the other hand, the maximum size of the space mesh which is permitted by the geometry of the problem is $\Delta\xi = \Delta\eta = \frac{\ell}{2}$.

We carried out the finite difference solution of problem (3.2), which is a problem in one space-dimension, at an interval of $\frac{\ell}{2} = .006$. After 30 time steps the numerical solution agreed with the exact solution (3.5) to within 3 percent.

We made the same comparison with the same good results for the two-dimensional problem which is suggested by the situation along the axis of our cylinder. That is, a bar of steel of length ℓ and a bar of constantin of length 4ℓ are joined together. The exposed steel is heated. The equations governing the flow of heat are:

$$\text{(7.1)}\quad \underline{\text{P.D.E.}} \quad \begin{cases} u_{\xi\xi} = u_\tau; & 0 \le \xi \le \ell, \quad \tau \ge 0, \\ u_{\xi\xi} = 2u_\tau; & \ell \le \xi \le 5\ell, \quad \tau \ge 0. \end{cases}$$

$$\text{(7.2)}\quad \underline{\text{B.C.}} \quad \begin{cases} u_\xi = u - 1 & \text{at} \quad \xi = 0, \\ u|_{\text{left}} = u|_{\text{right}} & \text{at} \quad \xi = \ell, \\ u_\xi|_{\text{left}} = \frac{1}{2}\, u_\xi|_{\text{right}} & \text{at} \quad \xi = \ell, \\ u_\xi = 0 & \text{at} \quad \xi = 5\ell. \end{cases}$$

$$\text{(7.3)}\quad \underline{\text{I.C.}} \qquad u = 0 \quad \text{at} \quad \tau = 0.$$

The exact "Fourier series" solution* of this problem is:

---

*The method of obtaining the solution is standard and is explained for simpler problems in H. S. Carslaw, Conduction of Heat in Solids, pp. 74-79.

$$u = 1 + \sum_{n=1}^{\infty} \beta_n \chi_n(\xi) e^{-\delta_n^2 \tau}, \tag{7.4}$$

where δ_n is the nth positive root of

$$\delta_n = \frac{\sqrt{2} \cot 4\sqrt{2}\,\ell\delta_n - \tan \ell\delta_n}{1 + (\tan \ell\delta_n)(\sqrt{2} \cot 4\sqrt{2}\,\ell\delta_n)},$$

$$\chi_n(\xi) = \delta_n \cos \delta_n\xi + \sin \delta_n\xi; \qquad 0 \le \xi \le \ell,$$

$$\chi_n(\xi) = A_n \cos \sqrt{2}\,\delta_n\xi + B_n \sin \sqrt{2}\,\delta_n\xi; \qquad \ell \le \xi \le 5\ell,$$

with

$$A_n = B_n \cot 5\sqrt{2}\,\ell\delta_n,$$

and

$$B_n = \frac{\sqrt{2}[\cos \ell\delta_n - \delta_n \sin \ell\delta_n]}{\cos \sqrt{2}\,\ell\delta_n - (\cot 5\sqrt{2}\,\ell\delta_n)(\sin \sqrt{2}\,\ell\delta_n)}$$

while

$$\beta_n = -\frac{\frac{1}{\delta_n}}{[\frac{\ell}{2}(\delta_n^2 + 1) + \frac{1}{2}] + 2\ell B_n^2[\cot^2 5\sqrt{2}\,\ell\delta_n + 1]}.$$

A rough estimate of the time for solving (3.6) on the Mark I at an interval of $\Delta\xi = .006$ was one month. Since $\Delta\xi = .006$ was satisfactory in the two two-dimensional problems that we had considered, we decided that it was reasonable to use $\Delta\xi = .006$ for problem (3.6). However, it was only after we had obtained the results of the Mark I computation that we were certain that we had chosen a fine enough mesh.

Sec. 8 Analysis of numerical results

We decided to carry out the computation until the difference between the temperature at the hot junction and the temperature at the cylindrical wall at $\xi = \ell$ was less than 1 percent. The calculations were therefore continued for 3000 time steps. This corresponds to a time less than .08 seconds. In other words, the large perturbation of temperature occurs only in the beginning.

We print the values of the temperature u, for several times τ, in Table I (pages 33-42). Before we discuss the perturbation of temperature produced by the thermocouple, we analyze the results with another view in mind. That is, we compare the temperature along the cylindrical wall, $\eta = 5\ell$, with that obtained from the solution (3.5) of the problem (3.2) in one space-dimension.

We expect the temperature at the cylindrical wall to be higher than that which would obtain without the thermocouple, because of the fact that constantin has a smaller heat conductivity and heat capacitance than steel. In keeping with this observation is the fact that the temperature given by (3.5) is less than 4 percent lower than the temperature of the outer wall of the cylinder given by the Mark I computations (throughout the 3000 time steps). A summary of these results will be found in Table II (page 23). In addition to this convincing evidence of the accuracy of the calculations, we introduce the comparison between the temperature along $\eta = 0$ (i.e. the axis of the cylinder) and that found in the corresponding one space-dimension problem (7.1). Again the temperatures are very close and are within 3 percent of each other after 500 time steps.

The remarkable agreement which we have found between the solutions (3.5), (7.4) and the finite difference calculations of the Mark I allows us to draw the following conclusion:

We may replace the study of the three-dimensional problem with its complicated boundary conditions by the study of the simple two-dimensional problems (3.2) and (7.1), which have been solved explicitly in (3.5) and (7.4). This permits us to vary the physical parameters without repeating the problem on high-speed calculators.

The situation we have just met is probably a typical example of what we may expect in the near future when we use the newly developing high-speed computing machines. That is, the various simple approximations which have been developed by the mathematician will be compared with the results of one or several solutions computed by the automatic calculator. The simple mathematical solutions which give good agreement with the numerical calculations will then be available for further study of the physical problem.

Sec. 9 The perturbation of temperature

We find that the average temperature over the hot face of the constantin is 32 percent higher than that at the cylindrical wall (at the same distance from the hot wall) when $\tau = .0009$. This difference of temperatures decreases quickly with time and is less than 1 percent at $\tau = .027$. We have drawn a graph of the ratio of the temperature at the cylindrical wall to that over the hot face of the constantin (Figure 3). We also present a graph of the ratio of the temperature given by (3.5) at $\xi = \ell$ to that given by (7.4) at $\xi = \ell$ as a function of time, τ. It is clear that initially there is a considerable perturbation of temperature which quickly decreases. If we are interested in measuring the temperature of the steel during the short period of time when the rocket grain is burning, we must take into account the perturbation created by the thermocouple. That is, we must multiply the temperature which the thermocouple yields by a factor which increases in time, and which is approximately given by the above described ratios.

Perturbation Factor = Ratio $\frac{\text{of temperature at cylindrical wall}}{\text{to temperature at hot junction}}$ = P

(as a function of dimensionless time τ)

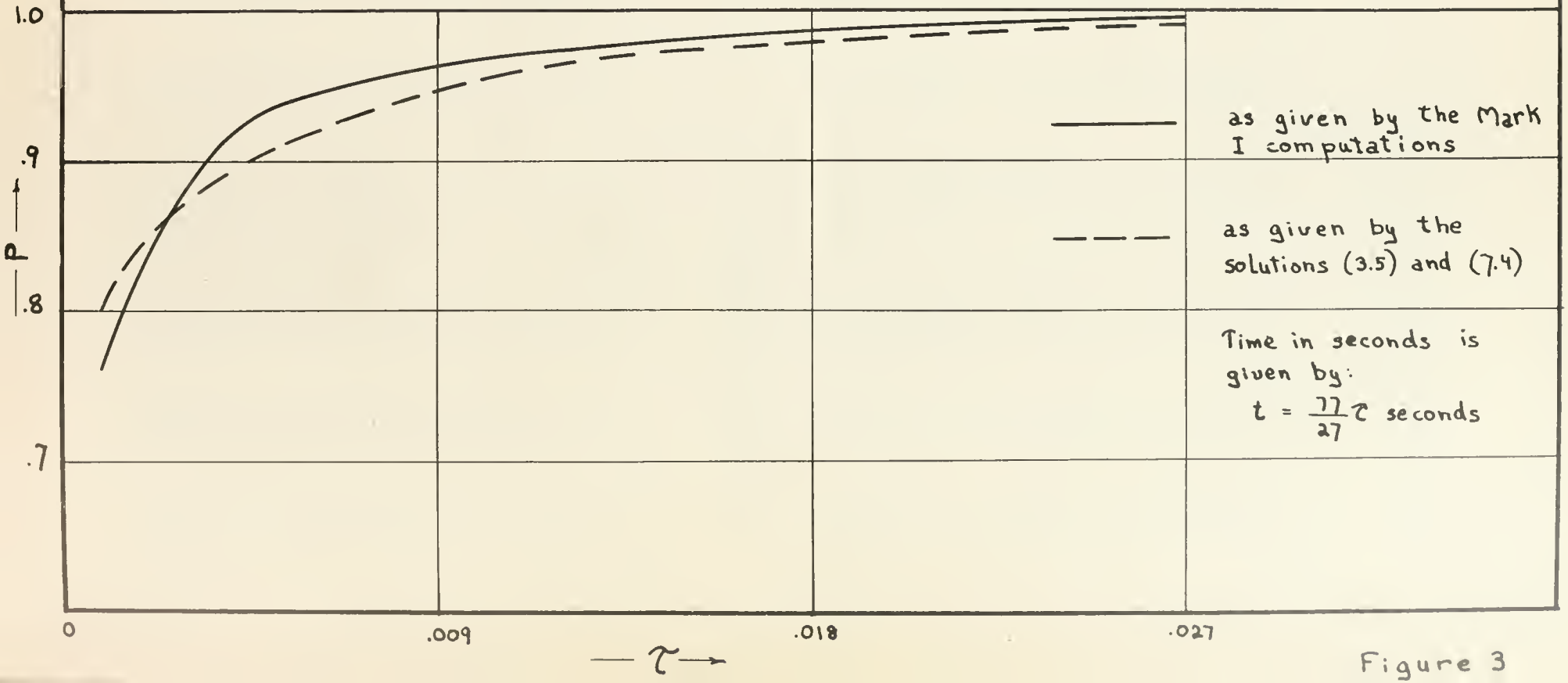

Figure 3

Table II

Temperature given by (3.5) and the corresponding temperature given by Mark I at the cylindrical wall

$\eta = .060$

τ	Method of Calculation	ξ					
		.000	.006	.012	.018	.024	.060
.0009	Mark I	.0336	.0281	.0232	.0191	.0155	.0060
	(3.5)	.0331	.0276	.0228	.0187	.0152	.0059
.0045	Mark I	.0918	.0866	.0819	.0778	.0742	.0640
	(3.5)	.0892	.0841	.0794	.0753	.0717	.0618
.0090	Mark I	.1585	.1537	.1494	.1456	.1422	.1328
	(3.5)	.1538	.1490	.1446	.1408	.1375	.1283
.0135	Mark I	.2203	.2159	.2119	.2083	.2052	.1965
	(3.5)	.2138	.2093	.2053	.2017	.1987	.1901
.0180	Mark I	.2776	.2735	.2698	.2665	.2636	.2555
	(3.5)	.2695	.2654	.2616	.2583	.2555	.2475
.0225	Mark I	.3306	.3268	.3233	.3203	.3176	.3102
	(3.5)	.3213	.3174	.3140	.3109	.3082	.3008
.0270	Mark I	.3797	.3761	.3730	.3701	.3677	.3608
	(3.5)	.3697	.3661	.3629	.3600	.3576	.3507

Temperature given by (7.4) and the corresponding temperature given by Mark I along the axis of the cylinder

$\eta = .000$

τ	Method of Calculation	ξ					
		.000	.006	.012	.018	.024	.060
.0009	Mark I	.0384	.0335	.0303	.0236	.0177	.0028
	(7.4)	.0382	.0329	.0284	.0211	.0151	.0024
.0045	Mark I	.0958	.0910	.0875	.0792	.0721	.0523
	(7.4)	.0979	.0927	.0882	.0800	.0729	.0531
.0090	Mark I	.1622	.1578	.1544	.1465	.1395	.1203
	(7.4)	.1613	.1565	.1522	.1446	.1379	.1196
.0135	Mark I	.2238	.2196	.2165	.2091	.2027	.1847
	(7.4)	.2201	.2157	.2117	.2046	.1985	.1814
.0180	Mark I	.2803	.2762	.2724	.2656	.2597	.2434
	(7.4)	.2749	.2717	.2670	.2604	.2547	.2388
.0225	Mark I	.3331	.3293	.3257	.3194	.3138	.2984
	(7.4)	.3258	.3219	.3185	.3124	.3070	.2922
.0270	Mark I	.3820	.3784	.3752	.3692	.3641	.3498
	(7.4)	.3731	.3695	.3663	.3606	.3557	.3420

Sec. 10 The machine computation

The computing task

The factors affecting the decision as to the size of the mesh-width were discussed in Section 7. With the mesh-width fixed at $\Delta\xi = \Delta\eta = .006$, the computational problem admitted the following description: to solve simultaneously 121 linear recurrence relations and to obtain the solution at the nth recurrence. The time number n at which to terminate the solution was fixed, as the computation proceeded, at 3000.

Storage

This raised an immediate storage problem. Since the values for the time τ_1 were to be computed from the values for the time τ_o, simultaneous storage was needed for more than the 11^2 values which represent the solution for each round: the number $u^o_{o,o}$ cannot be discarded as soon as $u^1_{o,o}$ has been computed, for it will be needed for the computation for $u^1_{o,1}$. Some overlap was required; by a suitable ordering of the points it was possible to reduce the overlap to twelve points.

Since the full accuracy of the calculator was not required, it was possible to store two quantities in each (24-column) storage register. (This is done very conveniently in Mark I by means of the "multiple in-out counter" designed especially for such use. Further circuits were designed to make the use of this counter more effective during the running of this problem. These changes were permanently incorporated in the machine.) This parallel storage reduced the computation time by approximately 25 percent through "parallel multiplication," in which the two quantities in a register are multiplied simultaneously by a third quantity.

The nature of the problem permits further reduction of the storage requirements. The values at the points (0,10), (10,10), and (10,2) are not needed for the computation and, therefore, need not be computed. Also, if this coarse mesh is used, the expression for the temperature at a point on the lower edge of the nonconductor is

$$u^{o}_{o,o} = \frac{1}{3}[4u^{o}_{o,-1} - u^{o}_{o,-2}]$$
$$= \frac{1}{3}[4u^{o}_{o,-1} - u^{o}_{o,o}] \qquad \text{(by symmetry)},$$

so that for such a point

$$u^{o}_{o,o} = u^{o}_{o,-1}$$

and computation for the temperature at the points (3,1), (4,1), $\cdots$, (10,1) can be omitted. Then for each time τ the temperature was computed at each of 110 points. This computation was contained well within the available doubled-up storage capacity. If it were not, another reduction could have been used. The temperature of a boundary point (to fix the ideas, say a point of the left-hand boundary) is given by

$$u^{o}_{o,o} = f(u^{o}_{1,o}, u^{o}_{2,o})$$

and is required only to compute

$$u^{1}_{1,o} = g(u^{o}_{o,o}, u^{o}_{2,o}, u^{o}_{1,1}, u^{o}_{1,-1}).$$

Thus, at the outset,

$$u^{1}_{1,o} = g(f(u^{o}_{1,o}, u^{o}_{2,o}), u^{o}_{2,o}, u^{o}_{1,1}, u^{o}_{1,-1})$$
$$= h(u^{o}_{1,o}, u^{o}_{2,o}, u^{o}_{1,1}, u^{o}_{1,-1}),$$

and the explicit computations at each of the thirty-six boundary points may be omitted. By means of such devices

the computation for the next finer mesh of 21^2 apparent points can be carried out by computing at only 330 points.

Special properties of $u^{\tau}_{\xi,\eta}$

The temperature function has certain properties which can be exploited by the computer to set up the computation in simpler and shorter form. Some of these are immediately evident. For instance, from the finite difference equations it is clear that isolated small errors in the computation are dissipated rapidly. Hence a difference check which might admit small machine errors can be regarded as adequate.

A more careful study of the effect of error appears in Section 6. From this study it can be seen that an error, even if systematically introduced, is damped, so that the effect of drop-off errors--which appear in every multiplication--is at worst imperfectly cumulative. Thus, although u is computed to only six decimal places, and the drop-off errors all occur in the same direction through 3000 rounds of computation, it was possible, nevertheless, to estimate the tabulated function at $\tau = 3000$ to be correct to within 5×10^{-4}.

Some special properties of the solution function are not immediately obvious: for example, it might not be at once apparent from the physical hypotheses that $u_\eta \leqq 0$ whenever $\eta > 2$. However, this is true, and this knowledge could have been used to advantage in arranging the coding. To give another example: it might be expected that for many of the interior points the contribution (in equation 4.1) due to the term

$$\frac{\Delta\eta}{8\eta_o}[u^o_{o,1} - u^o_{o,o}]$$

would be always less than 10^{-6} and so could be omitted from the calculation. Unfortunately, it is not possible

to determine those points to which this applies until a tabulation of $u(\xi,\eta,\tau)$ has been completed. However, if it is desired to repeat the computation on a finer mesh, the table already obtained may be used.

Checks

The computations were checked by verifying that the first differences with respect to time τ did not exceed a certain tolerance. Mathematically, the check condition was

$$u^1_{\xi,\eta} < u^o_{\xi,\eta} + e, \tag{10.1}$$

where the tolerance e was taken independent of ξ,η but dependent on τ (e diminishes as τ increases). At first it was not known whether the first differences $u^1_{\xi,\eta} - u^o_{\xi,\eta}$ were distributed sufficiently uniformly for such use. In this respect the resultant behavior was smoother than could have been hoped. In fact, it was so smooth that the double inequality

$$u^o_{\xi,\eta} + e_1 < u^1_{\xi,\eta} < u^o_{\xi,\eta} + e_2 \tag{10.2}$$

could have been used, with e_1, e_2 dependent upon τ but not upon ξ,η. It should be noted that an error of deficiency of $u^1_{\xi,\eta}$ will pass the check based upon (10.1) but will be detected one round later. For

$$u^1 - E < u^o + e^o,$$

$$u^2 < u^1 + e^1,$$

but

$$u^2 \not< (u^1 - E) + e^1$$

if E is large. Thus, the check corresponding to (10.2) is implicit in the check corresponding to (10.1), but in less refined form.

For a further refinement of the checking procedure that is well adapted to machine computation, it can be required that the second differences do not exceed a certain tolerance. (The tolerance can be taken as a function of τ and independent of ξ,η.) Higher order differences might be useful in higher accuracy computations of this kind; for six-place computation, as in this case, second differences suit the purpose. Of course, the tolerance must be determined and successively decreased as the computation proceeds. In the range of computation covered, the second differences were less than 2×10^{-6} for all but the first 100 rounds.

The smooth behavior of the temperature function $u(\xi,\eta,\tau)$, as defined by the finite difference equations, presents certain dangers. For instance, consider a coding error which, by reading out of the wrong storage register, has the effect of replacing $u^{o}_{o,o}$ by a neighboring u-value $u^{o}_{o,1}$. An error of this kind could easily pass the more obvious checks and give a misleading impression as to the local behavior of the function, and suitable precautions had to be taken against the possibility of such errors.

Consider the corner point (0,10). It is possible to derive an algebraic expression for the temperature at such a point by treating it as a point on the upper boundary; alternatively, by treating it as a point on the left-hand boundary. The different algebraic expressions thus obtained are equivalent, as can be seen by writing them in terms of the temperatures at the points (1,9), (2,9), (1,8), (2,8). The same remark holds at the corner points (10,10) and (10,2). These algebraic identities can be used to provide additional checks on the computation. (A similar remark cannot be made for corner points at the junction end of the thermocouple.)

Changes in the equations

Certain of the equations of Section 4 cannot be used as they stand in the case of the 11^2 mesh. The equations which must be modified are those which give the temperature at the points (0,0), (1,0), (2,0), (0,1), (0,2), (1,1), (1,2), (2,2), as well as the points (3,1) through (10,1) discussed elsewhere in this section. Let x, y, z, u, v, w, r, s, t represent the temperature at the time τ at the first nine of these points, respectively, and a, b denote the respective temperatures at the same instant at the points (2,3), (2,4). Then by Section 4

$$r = R(s,t),$$

$$t = T(r,s,a,b).$$

The interdependence of r and t (which occurs only for this coarse mesh) makes some algebraic manipulation necessary before the computation can proceed. For example, if r is eliminated from the second of these equations,

$$t = F(R(s,t),s,a,b)$$

and this equation is solved for t, there results an equation of the form

$$t = G(s,a,b).$$

The interdependence at the other points can be treated similarly. The equations thus obtained were used in the computation in the form

$$z = k_1\left\{k_2[4(a + s) - b] - 4s - k_3\right\},$$

$$w = \lambda_1[\lambda_2 v + z - \lambda_3],$$

$$z = \mu_1[\mu_2(4c - d) + y - \mu_3],$$

where

$$k_1 = \frac{1}{6(3 + 2\Delta\xi) - 1},$$

$$k_2 = 3 - 2\Delta\xi,$$

$$k_3 = 2\Delta\xi,$$

$$\lambda_1 = \frac{3(3 + 2\Delta\xi)}{6(3 + 2\Delta\xi) - 1},$$

$$\lambda_2 = \frac{8(1 + \Delta\xi)}{3(3 + 2\Delta\xi)},$$

$$\lambda_3 = \frac{2\Delta\xi}{3(3 + 2\Delta\xi)},$$

$$\mu_1 = \frac{8(1 + \Delta\xi)}{3(1 + k_2/k_1)(3 + 2\Delta\xi) - 1},$$

$$\mu_2 = \frac{(k_2/k_1)(3 + 2\Delta\xi)}{8(1 + \Delta\xi)},$$

$$\mu_3 = \frac{\Delta\xi}{4(1 + \Delta\xi)}.$$

Once t, w, z have been obtained from these equations, r, u, and x can be obtained by the equations of Section 4. The interdependence that occurs at points like (3,1) leads to the simple result mentioned elsewhere in this section.

There is some difficulty at the right-hand boundary points for certain early time-values. The expression

$$u^1_{o,o} = \frac{1}{3}[4u^o_{-1,o} - u^o_{-2,o}]$$

is negative at the tenth stage where $u^o_{-1,o}$ is still zero but $u^o_{-2,o}$ has a positive value. A negative value for the temperature is not acceptable, and might be avoided by assuming an initial temperature of 10^{-6}. But this device requires further elaboration, for the drop-off error in

the multiplication $(3 \times 10^{-6}) \times 0.333333$ reduces the temperature to zero at the time and points in question. Also it should be noted that because of drop-off errors the computation shows a positive increment of heat at the points (10,n), not at the tenth round but at the twelfth. Of the several available ways to avoid accepting negative temperatures, probably the simplest and best is to replace negative temperatures by zero whenever they occur. This is easily done by means of a subsidiary sequence control.

This difficulty occurs also at several points on the line of symmetry $\eta = 0$ and it persists on this line through time 30.

Recording and output

In this computation it was possible to print each u-value as it was computed. However, the printing was done in the order of computation, which was an arbitrary arrangement imposed by the necessity of keeping the number of u-values in storage at any one time at a minimum. In a large-scale calculating machine specifically designed for numerical solution of partial differential equations by finite differences, a much larger storage capacity would be desirable. For computations on a finer mesh, a storage capacity of several thousand quantities might be required. Such a computation would proceed too rapidly for the typewriters; it would be necessary to print out enough of the computed data to enable the mathematician to remain informed as to the state of the computation. As the computation proceeded on Mark I, tabulated numbers were preserved on punched cards, so that the computed temperature table for any instant could be obtained from the cards and printed according to a natural arrangement. Of course, a corresponding procedure

could be available on a higher speed machine solving a similar problem.

It is to be emphasized that a problem which involves the simultaneous solution in a short time of a large number of simultaneous recurrence relations cannot be handled effectively by allotting the computations to a large group of computers using hand machines, because the calculations for round τ_n must be completed before the calculations for round τ_{n+1} are begun. The numerical method used here is thus made practicable only by large-scale high-speed calculating machines.

Table I

The distribution of the dimensionless temperature, u, at various times

$\tau = .0009$

.060	.0335	.0281	.0232	.0191	.0155	.0125	.0102	.0083	.0071	.0063	.0060
.054	.0336	.0281	.0233	.0191	.0155	.0125	.0102	.0083	.0071	.0063	.0060
.048	.0337	.0282	.0233	.0191	.0156	.0126	.0102	.0084	.0071	.0063	.0061
.042	.0338	.0283	.0235	.0192	.0156	.0126	.0102	.0084	.0071	.0063	.0061
.036	.0341	.0286	.0237	.0194	.0158	.0127	.0103	.0084	.0071	.0064	.0061
.030	.0345	.0290	.0240	.0197	.0159	.0128	.0104	.0085	.0072	.0064	.0061
.024	.0351	.0296	.0246	.0200	.0162	.0130	.0104	.0085	.0072	.0064	.0061
.018	.0360	.0305	.0256	.0205	.0164	.0131	.0105	.0086	.0072	.0064	.0061
.012	.0370	.0319	.0280	.0206	.0164	.0131	.0105	.0086	.0072	.0064	.0061
.006	.0378	.0332	.0310	.0236	.0177	.0128	.0089	.0061	.0042	.0031	.0028
.000	.0384	.0335	.0303	.0236	.0177	.0128	.0089	.0061	.0042	.0031	.0028
η / ξ	.000	.006	.012	.018	.024	.030	.036	.042	.048	.054	.060

To convert from dimensionless units use:

$T = 3000u \quad {}^{\circ}C$ $\qquad$ $x = \frac{11}{18}\xi$ centimeters

$t = \frac{77}{27}\tau$ seconds $\qquad$ $r = \frac{11}{18}\eta$ centimeters

Table I (continued)

τ = .0018

.060	.0491	.0437	.0388	.0345	.0307	.0275	.0249	.0229	.0214	.0205	.0202
.054	.0491	.0437	.0388	.0345	.0307	.0275	.0249	.0229	.0214	.0205	.0202
.048	.0492	.0438	.0389	.0346	.0308	.0276	.0249	.0229	.0214	.0205	.0202
.042	.0494	.0439	.0390	.0347	.0309	.0276	.0250	.0229	.0214	.0206	.0203
.036	.0496	.0442	.0392	.0348	.0310	.0277	.0250	.0230	.0215	.0206	.0203
.030	.0500	.0445	.0395	.0351	.0311	.0278	.0251	.0230	.0215	.0206	.0203
.024	.0505	.0450	.0400	.0354	.0313	.0279	.0252	.0231	.0215	.0206	.0203
.018	.0513	.0459	.0410	.0358	.0315	.0280	.0252	.0231	.0216	.0206	.0203
.012	.0522	.0471	.0434	.0359	.0316	.0281	.0253	.0231	.0216	.0206	.0203
.006	.0528	.0483	.0459	.0373	.0308	.0254	.0210	.0177	.0153	.0139	.0134
.000	.0534	.0485	.0450	.0373	.0308	.0254	.0210	.0177	.0153	.0139	.0134
η / ξ	.000	.006	.012	.018	.024	.030	.036	.042	.048	.054	.060

Table I (continued)

$\tau = .0027$

.060	.0636	.0583	.0534	.0492	.0454	.0423	.0397	.0376	.0362	.0353	.0350
.054	.0636	.0583	.0535	.0492	.0455	.0423	.0397	.0376	.0362	.0353	.0350
.048	.0637	.0584	.0535	.0492	.0455	.0423	.0397	.0377	.0362	.0353	.0350
.042	.0639	.0585	.0537	.0494	.0456	.0424	.0398	.0377	.0362	.0353	.0350
.036	.0641	.0587	.0539	.0495	.0457	.0425	.0398	.0377	.0363	.0354	.0351
.030	.0644	.0591	.0542	.0497	.0459	.0426	.0399	.0378	.0363	.0354	.0351
.024	.0650	.0596	.0547	.0501	.0461	.0427	.0400	.0378	.0363	.0354	.0351
.018	.0658	.0604	.0556	.0504	.0462	.0428	.0400	.0379	.0363	.0354	.0351
.012	.0666	.0616	.0579	.0506	.0463	.0428	.0400	.0379	.0363	.0354	.0351
.006	.0673	.0627	.0603	.0514	.0445	.0387	.0340	.0304	.0278	.0262	.0257
.000	.0678	.0629	.0594	.0514	.0445	.0387	.0340	.0304	.0278	.0262	.0257
η / ξ	.000	.006	.012	.018	.024	.030	.036	.042	.048	.054	.060

Table I (continued)

$\tau = .0036$

.060	.0778	.0725	.0678	.0636	.0599	.0568	.0542	.0522	.0508	.0499	.0496
.054	.0778	.0726	.0678	.0636	.0599	.0568	.0542	.0522	.0508	.0499	.0496
.048	.0779	.0726	.0679	.0637	.0600	.0568	.0543	.0523	.0508	.0499	.0497
.042	.0781	.0728	.0680	.0638	.0601	.0569	.0543	.0523	.0508	.0500	.0497
.036	.0783	.0730	.0682	.0639	.0602	.0570	.0544	.0523	.0509	.0500	.0497
.030	.0786	.0733	.0685	.0641	.0603	.0571	.0544	.0524	.0509	.0500	.0497
.024	.0791	.0738	.0690	.0645	.0605	.0572	.0545	.0524	.0509	.0500	.0497
.018	.0799	.0746	.0699	.0648	.0607	.0573	.0546	.0524	.0509	.0500	.0497
.012	.0807	.0758	.0722	.0650	.0608	.0573	.0546	.0525	.0509	.0500	.0497
.006	.0814	.0769	.0745	.0653	.0583	.0523	.0474	.0437	.0410	.0394	.0388
.000	.0819	.0771	.0735	.0653	.0583	.0523	.0474	.0437	.0410	.0394	.0388
η / ξ	.000	.006	.012	.018	.024	.030	.036	.042	.048	.054	.060

Table I (continued)

$\tau = .0045$

.060	.0918	.0866	.0819	.0778	.0742	.0711	.0686	.0666	.0652	.0643	.0640
.054	.0918	.0866	.0819	.0778	.0742	.0711	.0686	.0666	.0652	.0643	.0640
.048	.0919	.0867	.0820	.0779	.0742	.0711	.0686	.0666	.0652	.0643	.0641
.042	.0920	.0868	.0821	.0780	.0743	.0712	.0686	.0667	.0652	.0644	.0641
.036	.0922	.0870	.0823	.0781	.0744	.0713	.0687	.0667	.0653	.0644	.0641
.030	.0926	.0874	.0826	.0783	.0746	.0714	.0688	.0667	.0653	.0644	.0641
.024	.0931	.0879	.0831	.0786	.0748	.0715	.0688	.0668	.0653	.0644	.0641
.018	.0938	.0887	.0840	.0790	.0749	.0716	.0689	.0668	.0653	.0645	.0642
.012	.0946	.0898	.0862	.0791	.0750	.0716	.0689	.0668	.0653	.0645	.0642
.006	.0953	.0908	.0884	.0792	.0721	.0660	.0611	.0573	.0545	.0529	.0523
.000	.0958	.0910	.0875	.0792	.0721	.066C	.0611	.0573	.0545	.0529	.0523
η / ξ	.000	.006	.012	.018	.024	.030	.036	.042	.048	.054	.060

Table I (continued)

$\tau = .009$

.060	.1585	.1537	.1494	.1455	.1422	.1393	.1370	.1352	.1339	.1331	.1328
.054	.1585	.1537	.1494	.1456	.1422	.1394	.1370	.1352	.1339	.1331	.1328
.048	.1586	.1538	.1495	.1456	.1422	.1394	.1370	.1352	.1339	.1331	.1328
.042	.1587	.1539	.1496	.1457	.1423	.1394	.1371	.1352	.1339	.1331	.1329
.036	.1589	.1541	.1498	.1458	.1424	.1395	.1371	.1353	.1339	.1331	.1329
.030	.1592	.1544	.1500	.1460	.1426	.1396	.1372	.1353	.1340	.1332	.1329
.024	.1597	.1549	.1505	.1463	.1427	.1397	.1373	.1354	.1340	.1332	.1329
.018	.1604	.1556	.1513	.1467	.1429	.1398	.1373	.1354	.1340	.1332	.1329
.012	.1611	.1567	.1534	.1468	.1430	.1398	.1373	.1354	.1340	.1332	.1329
.006	.1617	.1576	.1553	.1465	.1395	.1337	.1289	.1251	.1224	.1208	.1203
.000	.1622	.1578	.1544	.1465	.1395	.1337	.1289	.1251	.1224	.1208	.1203
η / ξ	.000	.006	.012	.018	.024	.030	.036	.042	.048	.054	.060

Table I (continued)

$\tau = .0135$

.060	.2203	.2159	.2119	.2083	.2052	.2026	.2004	.1987	.1975	.1968	.1965
.054	.2204	.2159	.2119	.2083	.2052	.2026	.2004	.1987	.1975	.1968	.1965
.048	.2204	.2160	.2120	.2084	.2053	.2026	.2004	.1987	.1975	.1968	.1966
.042	.2205	.2161	.2121	.2085	.2053	.2027	.2005	.1988	.1975	.1968	.1966
.036	.2207	.2163	.2122	.2086	.2054	.2027	.2005	.1988	.1976	.1968	.1966
.030	.2210	.2165	.2125	.2088	.2056	.2028	.2006	.1988	.1976	.1969	.1966
.024	.2215	.2170	.2129	.2091	.2057	.2029	.2006	.1989	.1976	.1969	.1966
.018	.2221	.2176	.2136	.2094	.2059	.2030	.2007	.1989	.1976	.1969	.1966
.012	.2228	.2186	.2156	.2095	.2059	.2030	.2007	.1989	.1977	.1969	.1966
.006	.2233	.2195	.2174	.2091	.2027	.1972	.1927	.1892	.1867	.1852	.1847
.000	.2238	.2196	.2165	.2091	.2027	.1972	.1927	.1892	.1867	.1852	.1847
η / ξ	.000	.006	.012	.018	.024	.030	.036	.042	.048	.054	.060

Table I (continued)

$\tau = .018$

.060	.2776	.2735	.2698	.2665	.2636	.2611	.2591	.2576	.2564	.2558	.2555
.054	.2776	.2735	.2698	.2665	.2636	.2612	.2591	.2576	.2565	.2558	.2556
.048	.2777	.2735	.2698	.2665	.2636	.2612	.2592	.2576	.2565	.2558	.2556
.042	.2778	.2736	.2699	.2666	.2637	.2612	.2592	.2576	.2565	.2558	.2556
.036	.2779	.2738	.2700	.2667	.2638	.2613	.2592	.2576	.2565	.2558	.2556
.030	.2782	.2740	.2702	.2668	.2639	.2614	.2593	.2577	.2565	.2558	.2556
.024	.2785	.2744	.2706	.2671	.2640	.2614	.2593	.2577	.2566	.2559	.2556
.018	.2790	.2749	.2712	.2673	.2641	.2615	.2594	.2577	.2566	.2559	.2556
.012	.2795	.2757	.2729	.2674	.2642	.2615	.2594	.2577	.2566	.2559	.2556
.006	.2799	.2762	.2737	.2656	.2597	.2548	.2507	.2475	.2453	.2439	.2434
.000	.2803	.2762	.2724	.2656	.2597	.2548	.2507	.2475	.2453	.2439	.2434
η / ξ	.000	.006	.012	.018	.024	.030	.036	.042	.048	.054	.060

Table I (continued)

$\tau = .0225$

.060	.3306	.3268	.3233	.3203	.3176	.3154	.3135	.3120	.3110	.3104	.3102
.054	.3306	.3268	.3234	.3203	.3176	.3154	.3135	.3121	.3110	.3104	.3102
.048	.3307	.3268	.3234	.3203	.3177	.3154	.3135	.3121	.3110	.3104	.3102
.042	.3308	.3269	.3235	.3204	.3177	.3154	.3136	.3121	.3110	.3104	.3102
.036	.3309	.3271	.3236	.3205	.3178	.3155	.3136	.3121	.3111	.3104	.3102
.030	.3311	.3273	.3238	.3206	.3179	.3155	.3136	.3121	.3111	.3104	.3102
.024	.3314	.3276	.3241	.3208	.3180	.3156	.3137	.3122	.3111	.3105	.3102
.018	.3319	.3281	.3247	.3211	.3181	.3157	.3137	.3122	.3111	.3105	.3103
.012	.3324	.3288	.3263	.3212	.3182	.3157	.3137	.3122	.3111	.3105	.3103
.006	.3327	.3293	.3269	.3194	.3138	.3091	.3053	.3023	.3002	.2989	.2984
.000	.3331	.3293	.3257	.3194	.3138	.3091	.3053	.3023	.3002	.2989	.2984
η / ξ	.000	.006	.012	.018	.024	.030	.036	.042	.048	.054	.060

Table I (continued)

$\tau = .027$

.060	.3797	.3761	.3730	.3701	.3677	.3656	.3638	.3625	.3615	.3609	.3608
.054	.3797	.3762	.3730	.3701	.3677	.3656	.3638	.3625	.3615	.3610	.3608
.048	.3798	.3762	.3730	.3702	.3677	.3656	.3639	.3625	.3615	.3610	.3608
.042	.3798	.3763	.3731	.3702	.3677	.3656	.3639	.3625	.3616	.3610	.3608
.036	.3800	.3764	.3732	.3703	.3678	.3657	.3639	.3626	.3616	.3610	.3608
.030	.3802	.3766	.3734	.3705	.3679	.3657	.3640	.3626	.3616	.3610	.3608
.024	.3805	.3769	.3737	.3707	.3680	.3658	.3640	.3626	.3616	.3610	.3608
.018	.3809	.3774	.3742	.3709	.3681	.3659	.3640	.3626	.3616	.3610	.3608
.012	.3813	.3780	.3757	.3710	.3682	.3659	.3641	.3626	.3616	.3610	.3608
.006	.3817	.3785	.3763	.3692	.3641	.3598	.3562	.3534	.3514	.3502	.3498
.000	.3820	.3784	.3752	.3692	.3641	.3598	.3562	.3534	.3514	.3502	.3498
η / ξ	.000	.006	.012	.018	.024	.030	.036	.042	.048	.054	.060

NYU
IMM-
164 c.1

NYU
IMM-
164 c.1

Isaacson
AUTHOR

Math. determination of trans[3]
TITLE
ient temp. in solids

DATE DUE	BORROWER'S NAME	ROOM NUMBER
MAR 28'61	J. Isaacson	
MAY 26 198	E. Isaacson	

N. Y. U. Institute of Mathematical Sciences
25 Waverly Place
New York 3, N. Y.

Date Due

MAR 28'61			
MAY 26 1987			
MAY 26 1987			

PRINTED IN U. S. A.